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Virtual distortions applied to structural modelling and sensitivity analysis. Damage identification testing example

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The paper presents so-called distortion approach to problems of structural (re)modelling and sensitivity analysis of linear systems under predetermined load. The load may be static or dynamic. In the first case the so-called Virtual Distortion Method (VDM) is used, while for dynamics the so-called Impulse Virtual Distortion Method (IVDM) have been developed, recently. All formulae are for structures composed of various finite elements. However, the very important question of distortion states in finite element is briefly discussed here for the case of 2D-beam finite element.

Key words: *virtual distortions, structural (re)modelling, sensitivity analysis, damage identification.*

1. Introduction

The *Virtual Distortion Method* (VDM) and its application to structural analysis, design and control of static structures were comprehensively described by Holnicki-Szulc [1], and Holnicki-Szulc and Gierlinski [2]. One of the preliminary descriptions which presents the new, *impulse* version of the method, i.e., the dedicated for dynamics of small vibrations *Impulse Virtual Distortion Method* (IVDM) may be found in paper by Holnicki-Szulc and Zieliński [3]. Thorough and complete description of the method gave Zieliński in [4]. In this work some new aspects of the classical VDM together with the description of object-oriented implementation of both VDM and IVDM were given.

This paper briefly presents some basic formulae and ideas of the distortion methods. Eventually, Sec. 7 brings a verification of some IVDM algorithms: the Impulse VDM is applied there to the damage identification problem based on the analysis of elastic wave propagation.

2. Some basic notions and ideas

Let us introduce briefly the following notions:

- the *virtual distortion* – the initial strain introduced in a structural element (its effect is analogous to the result of non-homogenous heating or geometric imperfection);
- the *unit-distortion* – the virtual distortion that would cause a unit-strain (of some kind) in an unconstrained element;
- the *compensating load* – the self-equilibrated load applied to the nodes of an element equivalent to the unit-distortion effect.

It is important to notice here that the virtual distortions, $\hat{\varepsilon}_i$, are to be used to model modifications of design parameters, p_s, that involves modification of structural stiffness, k_i.

We shall see that the most fundamental for VDM computations is so-called *influence matrix*. It groups static responses obtained for unit-distortions imposed successively on some of the structural elements. In practice, every column of the matrix is calculated for the adequate compensating load.

Zieliński [4] distinguished two kinds of the influence matrix. The first one is the *general influence matrix*, $\breve{D}_{\alpha i}$. The type of response grouped in this matrix is quite arbitral, i.e., depends on what is "in our interest". The only requirement is that it should be linearly dependent on $\hat{\varepsilon}_i$ (in practice, the response is just a linear combination of the generalized displacements of the finite element model degrees of freedom). Thus, having this matrix computed and knowing the distortions (which, for instance, model some structural modifications), we are able to calculate quickly the updated result (without any modification of the original structure):

$$f_\alpha = \overset{\text{L}}{f}_\alpha + \overset{\text{R}}{f}_\alpha = \overset{\text{L}}{f}_\alpha + \sum_i \breve{D}_{\alpha i}\, \hat{\varepsilon}_i\,. \tag{2.1}$$

Here, $\overset{\text{L}}{f}_\alpha$ is the original response to the load (i.e., linear term), while $\overset{\text{R}}{f}_\alpha$ is the response to distortions imposed on some elements and modelling some structural modifications (i.e., non-linear influence since distortions non-linearly depend on modifications).

A certain particular case of the general influence matrix will be the second distinguished type – we shall call it the *strain influence matrix*, D_{ij}. The responses grouped in this matrix are strains obtained in some distortion locations for successive unit-distortions imposed in these locations. Therefore, this is a square matrix, and all elements on its diagonal (i.e., D_{ii}) are in the interval $[0\,,1]$. Knowing this matrix we may calculate the general stresses and strains in some members of the structure:

$$S_i = \overset{\mathrm{L}}{S}_i + \overset{\mathrm{R}}{S}_i = k_i\, \overset{\mathrm{L}}{\varepsilon}_i + k_i \sum_j \big(D_{ij} - \delta_{ij}\big)\, \hat{\varepsilon}_j\,, \tag{2.2}$$

$$\varepsilon_i = \overset{\mathrm{L}}{\varepsilon}_i + \overset{\mathrm{R}}{\varepsilon}_i = \overset{\mathrm{L}}{\varepsilon}_i + \sum_j D_{ij}\, \hat{\varepsilon}_j\,. \tag{2.3}$$

Here, terms $\overset{\mathrm{L}}{S}_i$ and $\overset{\mathrm{L}}{\varepsilon}_i$ are the linear response to the load, while $\overset{\mathrm{R}}{S}_i$ and $\overset{\mathrm{R}}{\varepsilon}_i$ are the residual response for the imposed distortions that may model modifications and thus they are non-linear terms. However, the used above strain influence matrix tends to play more substantial role, since (as it will be shown below) it is used to determine these distortions.

3. Distortions in 2D-beam finite elements

Let us consider the well-known finite element of two-dimensional Bernoulli beam. Solving the eigenvalue problem for the stiffness matrix, $\mathbf{K}^{(e)}_{(6\times 6)}$, of this element provide us with six eigenvectors and corresponding six eigenvalues. Three of the eigenvalues are equal zero, and their eigenvectors describe three rigid motions of the 2D-beam element: two translations and one small-angle rotation. Thus, the remaining three eigenstates specify the complete *orthogonal basis of deformations*. These deformations should be assumed as the distortion states for this element. The form of these deformations and the compensating loads equivalent to the unit-distortions are presented in Fig. 1.

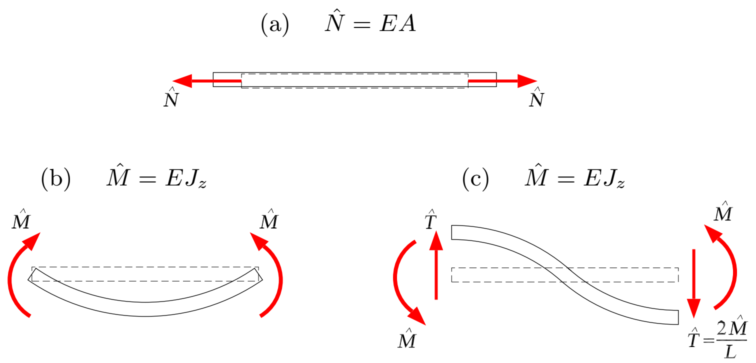

Figure 1. Orthogonal deformations (eigenstates) of the 2D-beam element assumed as the states of unit-distortions, and their compensating loads: (a) longitudinal tension, (b) pure bending, (c) asymmetric bending. (E – the Young modulus, A, J_z – the area and moment of inertia of cross-section).

The analogous approach of defining unit-distortion states as orthogonal, non-rigid eigenstates is proposed for other finite elements.

4. Distortion-based modelling of structural modifications in statics – the classical VDM

Let $\mathbf{p} = \big[p_s\big]$ be the *vector of structural parameters* and $\hat{\mathbf{p}} = \big[\hat{p}_s\big]$ – its modification. We demand that these parameters affect stiffness parameters, i.e.: $k_i = k_i(\mathbf{p})$, so after the modification we have: $\hat{k}_i = k_i(\hat{\mathbf{p}})$. We define the *parameters of stiffness modification* as: $\mu_i = \hat{k}_i/k_i$. Introducing the *vector of structural modification*: $\boldsymbol{\lambda} = \big[\lambda_s\big]$, where $\lambda_s = \hat{p}_s/p_s$, we may write that $\mu_i = \mu_i(\boldsymbol{\lambda})$. Now, let us consider the structure modified in some locations and the original structure but with some distortions imposed in these locations. We postulate that:

> *the structure modelled by distortions and the modified structure are identical in the sense of equality of their fields of general strains and stresses.*

General stresses and strains of the structure modelled by distortions are expressed by Eqs. (2.2) and (2.3). Thus, we substitute these equations to the constitutive relationship of the modified structure (under static load): $S_i = \hat{k}_i\,\varepsilon_i$, and after division by $k_i \neq 0$ and having used the definition for μ_i, we obtain:

$$\sum_j A_{ij}\,\hat{\varepsilon}_j = \big(1 - \mu_i(\boldsymbol{\lambda})\big)\,\overset{\scriptscriptstyle\mathrm{L}}{\varepsilon}_i \quad \text{where} \quad A_{ij} = \delta_{ij} - \big(1 - \mu_i(\boldsymbol{\lambda})\big)\,D_{ij}\,. \tag{4.1}$$

The above system of equations allows us to compute the distortions that model the modifications.

5. VDM applied to the sensitivity analysis of structures under static load

We want to determine how the specified response f_α is sensitive to structural modifications. This means that we want to be able to calculate the following gradient: $\frac{\partial f_\alpha}{\partial\lambda_s}(\boldsymbol{\lambda})$. From Eq. (4.1) it is evident that distortions $\hat{\varepsilon}_i$ depend on $\boldsymbol{\lambda}$, and so do responses $\overset{\scriptscriptstyle\mathrm{R}}{f}_\alpha$ (see Eq. (2.1)). Terms $\overset{\scriptscriptstyle\mathrm{L}}{f}_\alpha$ are the responses of the original structure (i.e., without any modifications) to the predetermined load and thus they don't depend on $\boldsymbol{\lambda}$. Therefore, we have that

$$\frac{\partial f_\alpha}{\partial\lambda_s}(\boldsymbol{\lambda}) = \frac{\partial \overset{\scriptscriptstyle\mathrm{R}}{f}_\alpha}{\partial\lambda_s}(\boldsymbol{\lambda}) = \sum_i \breve{D}_{\alpha i}\,\frac{\partial\hat{\varepsilon}_i}{\partial\lambda_s}(\boldsymbol{\lambda})\,. \tag{5.1}$$

The above relationship means that to compute the stiffness sensitivity we need to calculate the gradient of distortions, $\frac{\partial \hat{\varepsilon}_i}{\partial \lambda_s}(\boldsymbol{\lambda})$. To find this gradient we differentiate (4.1) with respect to λ_s, and as result we get the following linear system of equations:

$$\sum_j A_{ij} \frac{\partial \hat{\varepsilon}_j}{\partial \lambda_s} = -\frac{\partial \mu_i(\boldsymbol{\lambda})}{\partial \lambda_s} \left[\overset{\text{L}}{\varepsilon}_i + \sum_j D_{ij}\, \hat{\varepsilon}_j(\boldsymbol{\lambda}) \right]. \tag{5.2}$$

Here we should notice that before solving the above system of equations, we need to determine the distortions $\hat{\varepsilon}_i$ that model the modifications $\boldsymbol{\lambda}$. To this end we must solve the system (4.1). However, let us notice that the governing matrices for both systems, (4.1) and (5.2), are the same.

6. Distortion-based approach to the modelling of modifications and sensitivity analysis of structures under dynamic load – the recently developed Impulse VDM

For dynamic problems we need to introduce the time factor into the VDM. Thus, we assume that the virtual distortions depend on time. Therefore, the influence matrix is also time-dependent, and so now it will be 3-dimensional matrix. The Impulse VDM is based on so-called *method of impulse response function*. Thus, for IVDM we have so-called *impulse influence matrix*, since its every column is computed (using Newmark's method) for a unit-distortion applied as Dirac-like impulse in the initial instant, $\tau = 0$. In practice this impulse loads we perform in the initial conditions of Newmark's integrations of the homogenous equations of motion. This distortion-based approach proposed for structures under predetermined dynamic load allows to model the modifications of parameters that involve change of stiffness (modelling of mass modification is not yet implemented).

Thanks to the impulse influence matrices, $\breve{D}_{\alpha i}(t)$ and $D_{ij}(t)$, knowing distortion functions $\hat{\varepsilon}_i(t)$ we may compute, respectively: the actual dynamic response

$$f_\alpha(t) = \overset{\text{L}}{f}_\alpha(t) + \overset{\text{R}}{f}_\alpha(t) = \overset{\text{L}}{f}_\alpha(t) + \sum_{\tau=0}^{t} \sum_i \breve{D}_{\alpha i}(t-\tau)\, \hat{\varepsilon}_i(t)\,, \tag{6.1}$$

and time-varying strain functions in distortion locations

$$\varepsilon_i(t) = \overset{\text{L}}{\varepsilon}_i(t) + \overset{\text{R}}{\varepsilon}_i(t) = \overset{\text{L}}{\varepsilon}_i(t) + \sum_{\tau=0}^{t} \sum_j D_{ij}(t-\tau)\, \hat{\varepsilon}_j(\tau)\,. \tag{6.2}$$

However, it should be emphasized here that the *strain impulse influence matrix*, $D_{ij}(t)$, is mostly needed when we determine the distortion functions which model structural modifications $\boldsymbol{\lambda}$ that involve change of stiffness. To this end we solve, for every successive instant t, the following system of equations:

$$\sum_j A^0_{ij}\,\hat{\varepsilon}_j(t) = \begin{cases} \big(1-\mu_i(\boldsymbol{\lambda})\big)\,\overset{\mathrm{L}}{\varepsilon}_i(0) \quad \text{for} \;\; t=0\,, \\ \big(1-\mu_i(\boldsymbol{\lambda})\big)\Big[\overset{\mathrm{L}}{\varepsilon}_i(t) + \displaystyle\sum_{\tau=0}^{t-1}\sum_j D_{ij}(t-\tau)\,\hat{\varepsilon}_j(\tau)\Big] \quad \text{for} \;\; t>0\,, \end{cases} \tag{6.3}$$

where, fortunately, the governing matrix does not depend on t:

$$A^0_{ij} = \delta_{ij} - \big(1-\mu_i(\boldsymbol{\lambda})\big)\,D_{ij}(0)\,, \tag{6.4}$$

and thus is identical for all the systems. Please, notice that this matrix is in a way similar with the matrix $(4.1)_2$ used for determining static distortions.

The stiffness sensitivity of dynamic response to modification of structural parameters consists in determining the following gradient:

$$\frac{\partial f_\alpha}{\partial \lambda_s}(t,\boldsymbol{\lambda}) = \frac{\partial \overset{\mathrm{R}}{f}_\alpha}{\partial \lambda_s}(t,\boldsymbol{\lambda}) = \sum_{\tau=0}^{t}\sum_i \breve{D}_{\alpha i}(t-\tau)\,\frac{\partial \hat{\varepsilon}_i}{\partial \lambda_s}(t,\boldsymbol{\lambda})\,, \tag{6.5}$$

for which we need to compute the distortion gradient by solving, successively, for every instant t, the following systems of equations (for every s):

$$\sum_j A^0_{ij}\,\frac{\partial \hat{\varepsilon}_j}{\partial \lambda_s}(t) = \begin{cases} -\dfrac{\partial \mu_i(\boldsymbol{\lambda})}{\partial \lambda_s}\,\varepsilon_i(0,\boldsymbol{\lambda}) \quad \text{for} \;\; t=0\,, \\ -\dfrac{\partial \mu_i(\boldsymbol{\lambda})}{\partial \lambda_s}\,\varepsilon_i(t,\boldsymbol{\lambda}) + \big(1-\mu_i(\boldsymbol{\lambda})\big)\displaystyle\sum_{\tau=0}^{t-1}\sum_j D_{ij}(t-\tau)\,\frac{\partial \hat{\varepsilon}_j(\tau,\boldsymbol{\lambda})}{\partial \lambda_s} \\ \hfill \text{for} \;\; t>0\,. \end{cases} \tag{6.6}$$

The right-hand sides of these systems depend on the strain functions (6.2), and thus they depend on the distortions that are to be first determined from (6.3). Fortunately, the governing matrix for all the systems (6.6) is the same, and identical with the governing matrix for the system of equations (6.3). Thus, we need to perform the LU-decomposition only once.

Algorithms based on the approach briefly presented in this and the preceding Sections were thoroughly elaborated and described in [4].

7. Application of IVDM to the damage identification problem (a simple numerical experiment)

A numerical test for the IVDM sensitivity algorithm was performed regarding the problem of damage identification in the simple truss cantilever shown in Fig. 2. The identification base on the analysis of elastic wave propagation. All elements of the cantilever have the same material and section properties. It was assumed that the truss is excited by using activators at the free tip of the cantilever (elements Nos. 39 and 40), generating sine-wave excitations of identical amplitude and opposite phases. Elements Nos. 21 and 22 have sensors ready to read any change in the longitudinal strain of these bars (Fig. 2).

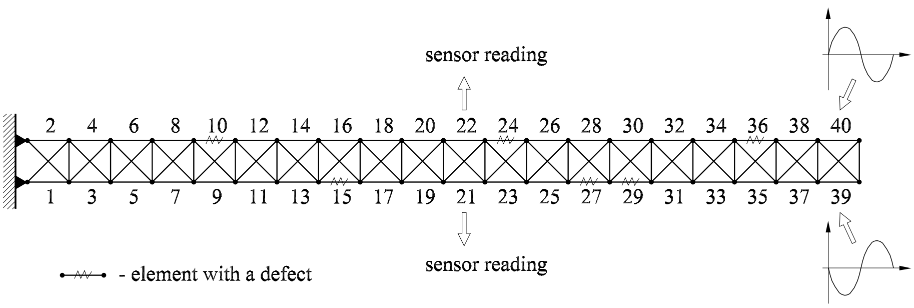

FIGURE 2. Truss cantilever.

For the assumed excitation signal the responses (i.e., sensor readings, see Fig. 3) were calculated numerically, first, for the case of the original structure (i.e., undamaged), and then for the truss with defects – it was assumed that several bars of the top and bottom flanges have defects reducing their structural stiffness. The defects were considered as thin cracks that reduce the effective area of cross-section but do not significantly affect the mass of element. They were modelled by assigning to every of the defected elements material of adequately reduced Young modulus. Values of defect intensities assumed for the damaged elements are given in Table 1. Localization of the

TABLE 1. Assumed defects intensities in some elements of the truss.

element No.:	$e =$	10	15	24	27	29	36
change of stiffness:	$\mu_{EA}^{(e)} = \frac{\hat{E}^{(e)}}{E^{(e)}} =$	0.8	0.7	0.6	0.6	0.7	0.5
defect intensity:	$1 - \mu_{EA}^{(e)} =$	20%	30%	40%	40%	30%	50%

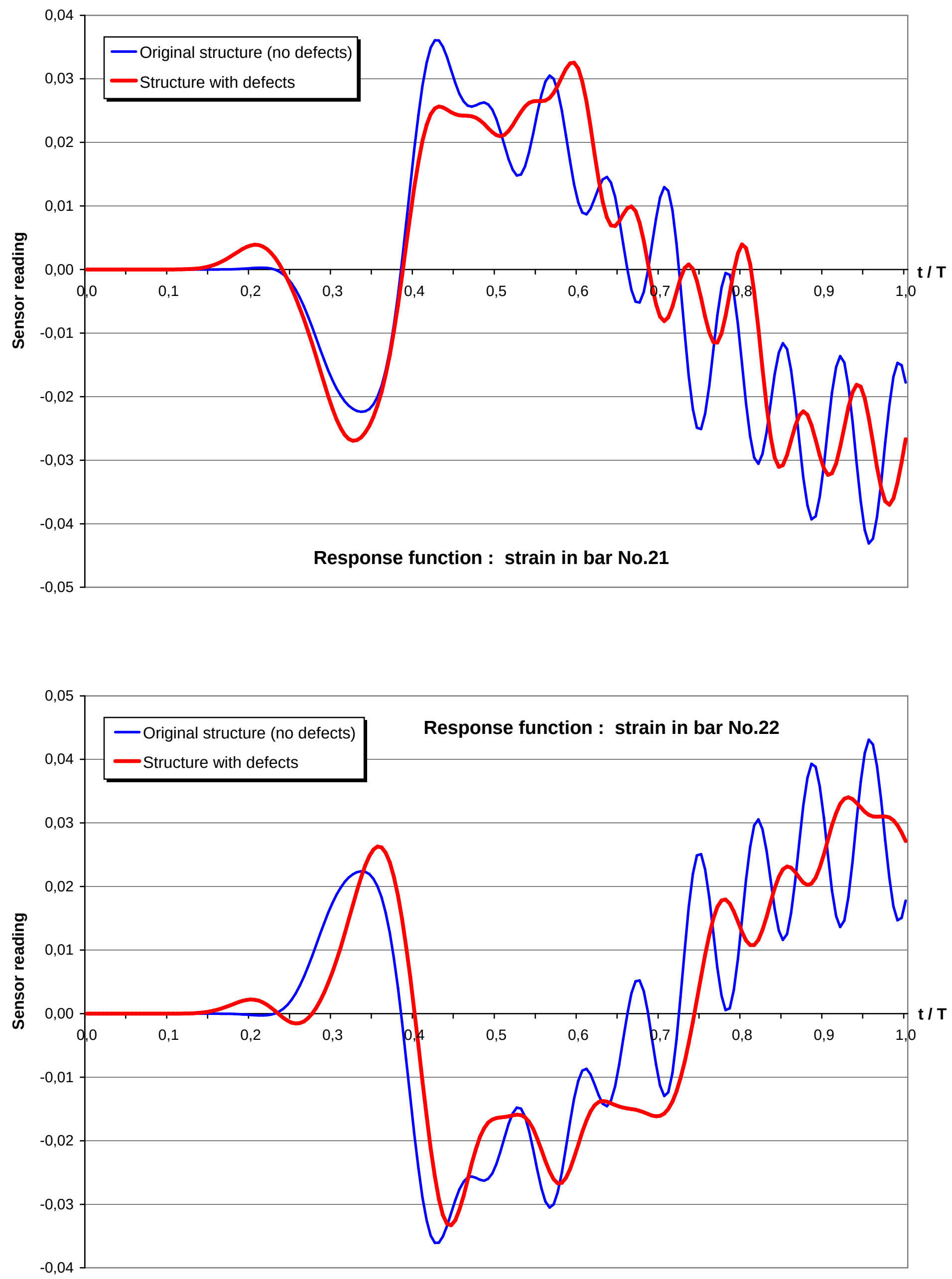

Figure 3. Forced responses of the truss structure with and without defects (i.e., "sensor readings" from elements Nos. 21 and 22).

defects is also marked in Fig. 2, while Fig. 3 presents the driven responses "read" from both sensors before and after the damage occurred.

In the examined damage identification problem we assumed that the defects are to be sought only in the elements of the both flanges of the truss cantilever but with the exception of the two elements with activators. Thus, we were seeking for defects only in the most probable flange elements Nos. 1 to 38 (see Fig. 2). The identification process was based on gradient approach where the objective function was defined as the squared difference between the responses obtained, respectively, for the original and damaged structure. For gradient calculations the IVDM algorithm for stiffness sensitivity of dynamically loaded structure was used. Actually, for the sake of comparison three independent damage identification processes were performed:

1. the identification using "readings" from the sensor situated in the element No. 21,
2. the identification using "readings" from the sensor situated in the element No. 22,
3. the identification using "readings" from the both sensors simultaneously.

In every of the identification processes calculations were performed in 40 iterations. The results are cumulatively presented in Fig. 4. The analysis of those graphs leads to the following observations and straightforward conclusions:

- Better results were obtained in the identification process using the sensor from the element No. 22 than in the process with the sensor from the element No. 21. However, because of the symmetry of the structure, localization of sensors and excitation, it is obvious that contrary situation shall happen for some other defect distributions.
- Obviously, the best results are from the identification making use of the readings obtained from both sensors, though localization of defects found using only one sensor (in the element No. 22) is sufficiently accurate (especially when considering the defects in the top flange).
- Defect identification (even using both sensors) cannot be fully accurate – the intensities of the identified defects are usually lower than the accurate ones, while some neighbouring elements exhibit false defects (though usually of comparatively small intensities). Thus, the identified defects tend to be fuzzy. This happens even for such a simple numerical experiment. Obviously, these small errors are not very disturbing but in more complicated tasks much important errors shall happen. This is because the solution of the identification problem is, generally speak-

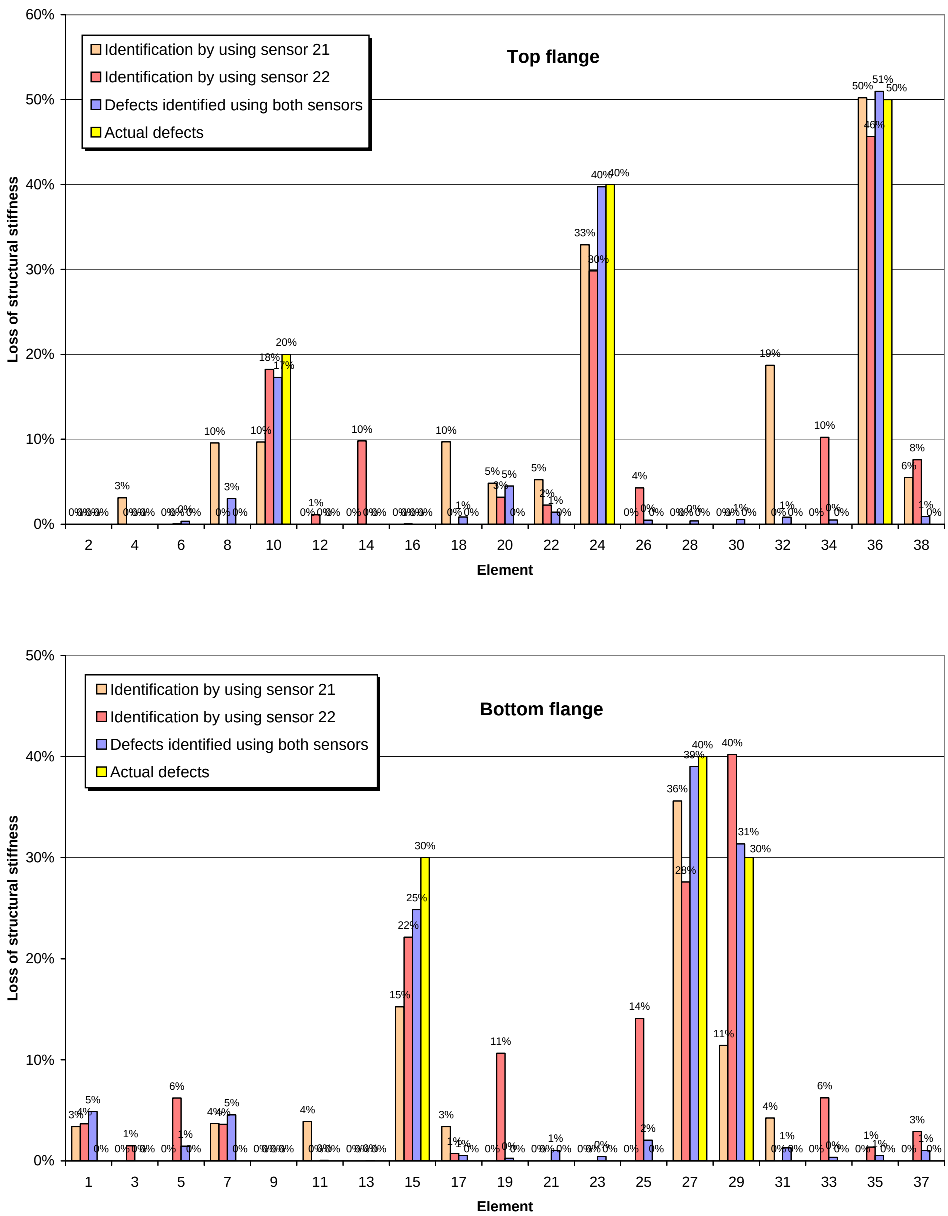

FIGURE 4. Defects identified in the elements of both flanges of the truss cantilever. In every position corresponding to an element from the top or bottom flange of the truss cantilever defect intensities identified independently in the three processes of identification are shown. The "actual" values of numerically-modelled defects are presented as well.

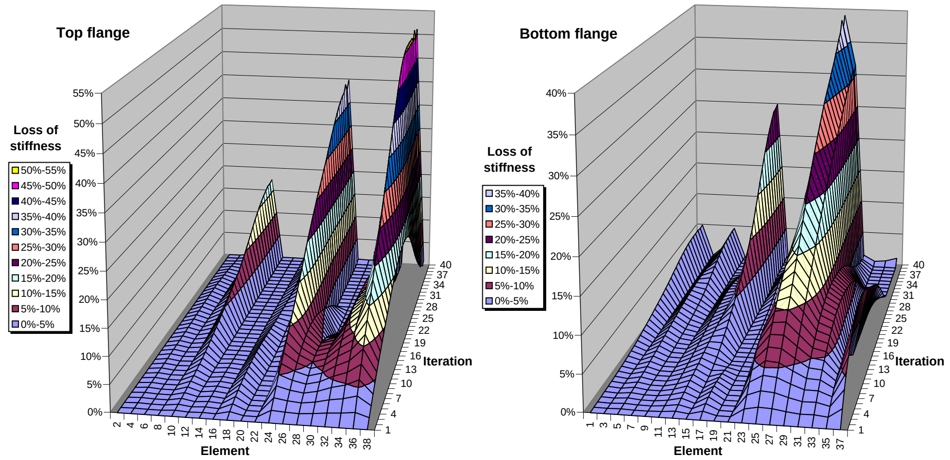

FIGURE 5. Histogram of the defect identification process (the identification using both sensors).

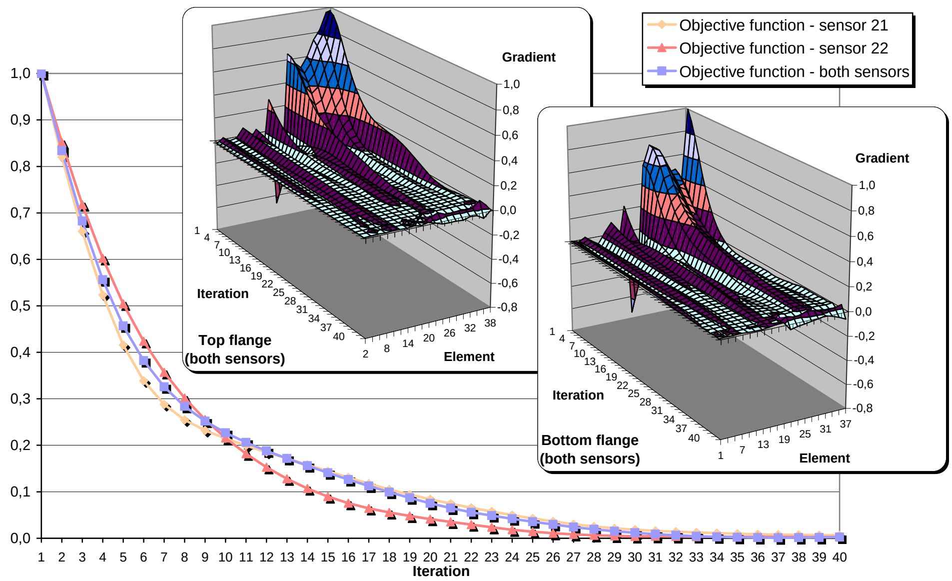

FIGURE 6. Objective function and its gradient computed in successive iterations of the damage identification process. Here, three objective functions for the three independent identification processes are presented, namely: the process using the sensor from the element No. 21, the process using the sensor from the element No. 22, and the process using the both sensors simultaneously. The presented graphs of the gradient of objective function was obtained from the process where both sensors were used. Values in the graphs are scaled by the corresponding maximal value obtained in the starting iteration.

ing, not unique; some systematic errors will play an important role as well (for example, an imperfect FE model, an inappropriate excitation signal, etc.).

- Small defects that cause little loss of stiffness might be difficult to identify especially in the presence of other bigger defects. Moreover, they can be hardly distinguished from the mentioned above fuzziness of identification results.

Figure 5 presents a histogram of the defect identification process where readings from both sensors were used, while Fig. 6 shows normalized plots of the corresponding objective functions obtained for all three identification processes. Figure 6 depicts also the gradient components of objective function obtained in successive iterations for the identification process with two sensors.

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